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MODULATION SPACES WITH SCALING SYMMETRY

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Abstract. We indicate how to construct a family of modulation spaces that have a scaling symmetry. We also illustrate the behavior of the Schrödinger multiplier on such function spaces.

1. Reconciling the modulation and dilation scalings

The Besov spaces and the modulation spaces are both examples of so-called decomposition spaces that stem from either a dyadic covering or uniform covering of the underlying frequency space. Roughly speaking, both of these classes of function spaces are defined by imposing appropriate decay conditions on a sequence of the form $\big\{\|\mathcal{F}^{-1}(\psi_k\widehat{f})\|_{L^p_x}\big\}_k$, where $\{\psi_k\}_{k\in\mathbb{Z}}$ is a dyadic partition of unity in the Besov case, while $\{\psi_k\}_{k\in\mathbb{Z}^d}$ is a uniform partition of unity in the case of the modulation spaces. Moreover, one can connect the two classes of spaces via the so-called α-modulation spaces of Gröbner [17] for $\alpha\in[0,1]$; the modulation spaces (and the Besov spaces, respectively) are obtained at the end-point: $\alpha=0$ (and $\alpha=1$, respectively). The α-modulation spaces are still decomposition spaces obtained via a "geometric interpolation" method that now asks for a covering $\{Q_k^\alpha\}_{k\in I}$ of the frequency space in which the sets satisfy a condition of the form $|Q_k^\alpha|\sim\langle\xi\rangle^\alpha$ for all $\xi\in Q_k^\alpha$; see [9].

In this paper, however, we are interested in a different connection between the two classes of function spaces than that alluded to above. It is clear from their definitions that the Besov spaces enjoy the dilation symmetry: $f(x)\mapsto f(2^kx)$, $k\in\mathbb{Z}$, while the modulation spaces enjoy the modulation symmetry: $f(x)\mapsto e^{2\pi ik\cdot x}f(x)$, $k\in\mathbb{Z}^d$. A natural question then is if there exists a class of function spaces that is sufficiently rich that would enjoy both such symmetries. As we shall see, reconciling the modulation and dilation symmetries is possible by considering scaled versions of the modulation spaces and then appropriately amalgamating them with a "good" vector weight. In many respects, such a class of function spaces would be more enticing to consider from the perspective of someone working in PDEs. Our note stems from these natural considerations and should be seen as a contribution to the general program of constructing new function spaces from given ones, as well as deriving the essential properties of the new spaces (i) from those of the modulation spaces on which they are rooted and (ii) from the characteristics of the weight, as proposed in [5].

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2. Definition of the $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$-spaces and their scaling property

2.1. **The $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$-spaces.** Let $p, q, r \in [1, \infty)$ be fixed. We denote by $\boldsymbol{w}$ a vector weight $\{w_j\}_{j\in\mathbb{Z}}$ with $w_j \geq 0$. Generally speaking, we aim to define a space which captures the modulations of all scales, while the choice of the weight $\boldsymbol{w}$ is made according to the task for which the space is needed.

Let us first recall next the definition of modulation spaces $M^{p,q}$; see [7, 8]. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\psi \subset Q_0 := [-1,1]^d \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}^d} \psi(\xi - k) \equiv 1. \tag{2.1}$$

Then, the modulation space $M^{p,q}$ is defined as the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{M^{p,q}} < \infty$, where the $M^{p,q}$-norm is defined by

$$\|f\|_{M^{p,q}(\mathbb{R}^d)} = \big\| \|\psi(D-k)f\|_{L^p_x(\mathbb{R}^d)} \big\|_{\ell^q_k(\mathbb{Z}^d)}. \tag{2.2}$$

Here, $\psi(D-k)f(x) = \int_{\mathbb{R}^d} \psi(\xi - k)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi$. Clearly, the support of ψ renders the domain of the integration of the previous integral to be $\xi \in Q_0 + k$, and in particular we have

$$f = \sum_{k\in\mathbb{Z}^d} \psi(D-k)f. \tag{2.3}$$

In what follows, for ease of notation and unless specifically stated otherwise, we assume that the underlying space is $\mathbb{R}^d$. Now, for a (fixed) *dyadic scale* $j \in \mathbb{Z}$, we define the scaled modulation space $M^{p,q}_{[j]}$ by the norm:

$$\|f\|_{M^{p,q}_{[j]}} := \big\| \|\psi_{j,k}(D)f\|_{L^p_x(\mathbb{R}^d)} \big\|_{\ell^q_k(\mathbb{Z}^d)},$$

where $\psi_{j,k}(D)$ is defined by

$$\psi_{j,k}(D)f(x) = \int_{\mathbb{R}^d} \psi(2^{-j}\xi - k)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi. \tag{2.4}$$

Namely, while the usual modulation spaces $M^{p,q}$ are adapted to the modulation symmetry of the unit scale, the $M^{p,q}_{[j]}$-spaces are adapted to the modulation symmetry of scale 2^j. When $j = 0$, we have $M^{p,q}_{[0]} = M^{p,q}$. Note also that, as in (2.3), we have

$$f = \sum_{k\in\mathbb{Z}^d} \psi_{j,k}(D)f \tag{2.5}$$

for all $j \in \mathbb{Z}$.

Now, in view of the decompositions (2.3) and (2.5), we have

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} w_j \psi_{j,k}(D)f$$

for any vector weight $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ with $\|\boldsymbol{w}\|_{\ell^1} = 1$. This motivates the following definition of the modulation spaces with scaling symmetry.

Definition 1. Let $1 \leq p, q, r < \infty$ and $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ be a vector weight with $w_j \geq 0$. We define the space $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}(\mathbb{R}^d)$ as the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}} < \infty$, where the $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$-norm is defined by

$$\|f\|_{\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}} := \big\|\|f\|_{M_{[j]}^{p,q}}\big\|_{\ell_j^r(\boldsymbol{w})} = \bigg(\sum_{j\in\mathbb{Z}} w_j^r \|f\|_{M_{[j]}^{p,q}}^r\bigg)^{\frac{1}{r}}. \tag{2.6}$$

Note that the definition of the space $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$ heavily depends on the choice of the weight $\boldsymbol{w}$ and we need to make sure that smooth functions belong to $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$ for suitable weights $\boldsymbol{w}$. We first claim that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is contained in $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$, provided that $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ is a "good" vector weight, that is, for some $\varepsilon > 0$ we have

$$w_j \lesssim \begin{cases} 2^{-\varepsilon j} & \text{if } j \geq 0, \\ 2^{-(\frac{d}{p'} - \frac{d}{q} - \varepsilon)j} & \text{if } j < 0. \end{cases} \tag{2.7}$$

Note that we allow some growth for $j < 0$, when $q > p'$. For $j \geq 0$, it suffices to impose summability of $\{w_j\}_{j\geq 0}$.

Let us first prove this claim. Let $\psi_{j,k}$ denote the multiplier of $\psi_{j,k}(D)$ defined in (2.4), corresponding to the smoothed version of $\mathbf{1}_{2^j(Q_0+k)}$. Now, for $j < 0$, it follows from Young's inequality that

$$\begin{aligned} \|\psi_{j,k}(D)f\|_{L_x^p} &\leq \|\mathcal{F}^{-1}(\psi_{j,k})\|_{L_x^p} \bigg\| \sum_{|i|\leq 1} \psi_{j,k+i}(D) f \bigg\|_{L_x^1} \\ &\sim 2^{j\frac{d}{p'}} \bigg\| \sum_{|i|\leq 1} \psi_{j,k+i}(D) f \bigg\|_{L_x^1}, \end{aligned}$$

where the implicit constant is independent of $k \in \mathbb{Z}^d$. By computing the ℓ_k^q-norm, we have

$$\begin{aligned} \|f\|_{M_{[j]}^{p,q}} &= \big\|\|\psi_{j,k}(D)f\|_{L_x^p}\big\|_{\ell_k^q(\mathbb{Z}^d)} \lesssim 2^{j\frac{d}{p'}} \big\|\|\psi_{j,k}(D)f\|_{L_x^1}\big\|_{\ell_k^q(\mathbb{Z}^d)} \\ &\leq 2^{j\frac{d}{p'}} \big\|\langle 2^j k\rangle^{-s} \|\psi_{j,k}(D)\langle\nabla\rangle^s f\|_{L_x^1}\big\|_{\ell_k^q(\mathbb{Z}^d)} \\ &\lesssim 2^{j\frac{d}{p'}} \|f\|_{W_x^{s,1}} \bigg(\sum_{k\in\mathbb{Z}^d} \frac{1}{\langle 2^j k\rangle^{sq}}\bigg)^{\frac{1}{q}} \\ &\lesssim 2^{jd(\frac{1}{p'} - \frac{1}{q})} \|f\|_{W_x^{s,1}} \end{aligned} \tag{2.8}$$

for any $s > \frac{d}{q}$. Here, $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$ and we used the Riemann sum approximation in the last step. Then, from (2.7) and (2.8), the contribution to the $\mathfrak{M}_{\boldsymbol{w}}^{p,q,r}$-norm for $j < 0$ is estimated by

$$\bigg(\sum_{j<0} w_j^r \|f\|_{M_{[j]}^{p,q}}^r\bigg)^{\frac{1}{r}} \lesssim \bigg(\sum_{j<0} 2^{\varepsilon jr}\bigg)^{\frac{1}{r}} \|f\|_{W_x^{s,1}} < \infty.$$

When $j \geq 0$, proceeding as above, we have

$$\|f\|_{M_{[j]}^{p,q}} \lesssim \|f\|_{W_x^{s,p}} \tag{2.9}$$

for any $s > \frac{d}{q}$. Hence, from (2.7) and (2.9), we obtain

$$\Big(\sum_{j\geq 0} w_j^r \|f\|^r_{M^{p,q}_{[j]}}\Big)^{\frac{1}{r}} \lesssim \Big(\sum_{j\geq 0} 2^{-\varepsilon j r}\Big)^{\frac{1}{r}} \|f\|_{W^{s,p}_x} < \infty.$$

We point out that the condition (2.7) is sharp. Let $j < 0$. Suppose that there exists $J \in \mathbb{N}$ such that $w_j \gtrsim 2^{-(\frac{d}{p'}-\frac{d}{q})j}$ for any $j \leq -J$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then, there exist $N \in \mathbb{N}$ and $k_0 \in \mathbb{Z}^d$ such that

$$|\widehat{f}(\xi)| \geq \frac{1}{2}\|\widehat{f}\|_{L^\infty}$$

for all $\xi \in 2^{-N}(Q_0+k_0)$ and $\widehat{f}$ essentially behaves like a constant on the cube $2^{-N}(Q_0+k_0)$. This in particular implies that

$$\|\psi_{j,k}(D)f\|_{L^p_x} \gtrsim \|\widehat{f}\|_{L^\infty} 2^{j\frac{d}{p'}},$$

for any $j \leq -N$ and $k \in \mathbb{Z}^d$ such that

$$2^j(Q_0+k) \subset 2^{-N}(Q_0+k_0). \tag{2.10}$$

Then, it is easy to see that $\|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} = \infty$ by simply computing the contribution from $j < 0$ and $k \in \mathbb{Z}^d$ satisfying (2.10). The condition (2.7) for $j \geq 0$ can also be seen to be essentially sharp by noting that

$$\|f\|_{M^{p,q}_{[j]}} \geq \|\psi_{j,0}(D)f\|_{L^p_x} \gtrsim \|f\|_{L^p_x}$$

for all sufficiently large $j \geq 0$.

Lastly, we wish to summarize the essential properties of the function spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ which bear a strong resemblance with those of the classical modulation spaces. The dilation property will be treated separately in the next subsection. See also Section 3 for a connection to other spaces stemming from PDEs.

Theorem 2.1. *Let $1 \leq p,q,r < \infty$ and assume that the vector weight $\boldsymbol{w}$ satisfies* (2.7). *Then, we have the following:*

- (i) $\mathcal{S}(\mathbb{R}^d) \subset \mathfrak{M}^{p,q,r}_{\boldsymbol{w}}(\mathbb{R}^d)$,
- (ii) $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}(\mathbb{R}^d) \subset M^{p,q}(\mathbb{R}^d) = M^{p,q}_{[0]}(\mathbb{R}^d)$,
- (iii) $\big(\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}(\mathbb{R}^d), \|\cdot\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}}\big)$ *is a Banach space,*
- (iv) $\mathfrak{M}^{p_1,q_1,r_1}_{\boldsymbol{w}}(\mathbb{R}^d) \subset \mathfrak{M}^{p_2,q_2,r_2}_{\boldsymbol{w}}(\mathbb{R}^d)$ *for $1 \leq p_1 \leq p_2$, $1 \leq q_1 \leq q_2$, and $1 \leq r_1 \leq r_2$.*

Remark 2.2. A natural and interesting question arises, concerning the duality of our function spaces. Let $1 \leq p,q,r < \infty$ and fix a vector weight $\boldsymbol{w} = \boldsymbol{w}(p,q) = \{w_j\}_{j\in\mathbb{Z}}$ satisfying the condition (2.7). Then, by setting

$$\langle f,g\rangle := \sum_{j\in\mathbb{Z}} w_j w_j' \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d} \big(\psi_{j,k}(D)f\big)(x)\big(\psi_{j,k}(D)g\big)(x)dx$$

for $f \in \mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$, Hölder's inequality yields

$$|\langle f,g\rangle| \leq \|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}}\|g\|_{\mathfrak{M}^{p',q',r'}_{\boldsymbol{w}'}},$$

where $\boldsymbol{w}' = \boldsymbol{w}'(p',q') = \{w_j'\}_{j\in\mathbb{Z}}$ is a vector weight satisfying the condition (2.7) with (p',q') in place of (p,q). We call such a vector weight $\boldsymbol{w}'$ a dual weight. This shows that $\bigcup_{\boldsymbol{w}'} \mathfrak{M}^{p',q',r'}_{\boldsymbol{w}'} \subset \big(\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}\big)'$, where the union is taken over all the dual weights $\boldsymbol{w}'$ (for a given pair (p,q)). It is, however, not clear to us how to determine the actual dual space $\big(\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}\big)'$ in this formulation.

Let us discuss an alternative approach. Let m be a counting measure on $\mathbb{Z}$ and write (2.6) as

$$\|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} := \|\psi_{j,k}(D)f\|_{L^r(\mathbb{Z},w_j^r dm)\ell^q_k(\mathbb{Z}^d)L^p_x(\mathbb{R}^d)}.$$

Then, a duality pairing may be given by

$$(f,g) := \int \bigg(\sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \big(\psi_{j,k}(D)f\big)(x)\big(\psi_{j,k}(D)g\big)(x)dx \bigg) w_j^r dm$$

In this case, Hölder's inequality (viewing $w_j^r dm$ as a measure on $\mathbb{Z}$) yields

$$|(f,g)| \le \|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} \|g\|_{\mathfrak{M}^{p',q',r'}_{\boldsymbol{w}'}},$$

where $\boldsymbol{w}' = \{w_j^{r-1}\}_{j\in\mathbb{Z}}$. This shows $\mathfrak{M}^{p',q',r'}_{\boldsymbol{w}'} \subset \big(\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}\big)'$ for this particular weight $\boldsymbol{w}'$. Note that this weight $\boldsymbol{w}'$ may not satisfy the condition (2.7) unless we impose a further assumption on $\boldsymbol{w}$.

2.2. **Scaling property.** Let us now investigate the scaling property of the new modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$. Define a translation operator τ on a vector weight $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ by setting $(\tau\boldsymbol{w})_j = w_{j+1}$. For $k \in \mathbb{N}$, write τ^k (and τ^{-k}, respectively) for τ composed with itself k times (and τ^{-1} composed with itself k times, respectively). Let us fix a dyadic scale $\lambda = 2^{j_0}$ for some $j_0 \in \mathbb{Z}$ and set $f_\lambda(x) = f(\lambda x)$. First, note that

$$\begin{aligned}
\big(\psi_{j,k}(D)f_\lambda\big)(x) &= \int_{\xi\in 2^j(Q_0+k)} \psi(2^{-j}\xi - k)\lambda^{-d}\widehat{f}(\lambda^{-1}\xi)e^{2\pi i x\cdot\xi}\, d\xi \\
&= \int_{\xi\in 2^{j-j_0}(Q_0+k)} \psi(2^{-j+j_0}\xi - k)\widehat{f}(\xi)e^{2\pi i\lambda x\cdot\xi}\, d\xi \\
&= \big(\psi_{j-j_0,k}(D)f\big)(\lambda x).
\end{aligned}$$

Hence, we have

$$\|\psi_{j,k}(D)f_\lambda\|_{L^p_x} = \lambda^{-\frac{d}{p}}\|\psi_{j-j_0,k}(D)f\|_{L^p_x} = 2^{-j_0\frac{d}{p}}\|\psi_{j-j_0,k}(D)f\|_{L^p_x}.$$

Therefore, we obtain

$$\begin{aligned}
\|f_\lambda\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} &= \big\|\|f_\lambda\|_{M^{p,q}_{[j]}}\big\|_{\ell^r_j(\boldsymbol{w})} \\
&= \Big\|\big\|\|\psi_{j,k}(D)f_\lambda\|_{L^p_x(\mathbb{R}^d)}\big\|_{\ell^q_k(\mathbb{Z}^d)}\Big\|_{\ell^r_j(\boldsymbol{w})} \\
&= 2^{-j_0\frac{d}{p}}\Big\|\big\|\|\psi_{j-j_0,k}(D)f\|_{L^p_x(\mathbb{R}^d)}\big\|_{\ell^q_k(\mathbb{Z}^d)}\Big\|_{\ell^r_j(\boldsymbol{w})} \\
&= 2^{-j_0\frac{d}{p}}\Big\|\big\|\|\psi_{j,k}(D)f\|_{L^p_x(\mathbb{R}^d)}\big\|_{\ell^q_k(\mathbb{Z}^d)}\Big\|_{\ell^r_j(\tau^{j_0}\boldsymbol{w})} \\
&= \lambda^{-\frac{d}{p}}\|f\|_{\mathfrak{M}^{p,q,r}_{\tau^{j_0}\boldsymbol{w}}}.
\end{aligned}$$

Note that the vector weight on the right-hand side is now given by $\tau^{j_0}\boldsymbol{w}$. Namely, in the general case, the scaling has an effect of translating the weight by $\log_2\lambda$.

Let us now assume further that the vector weight $\boldsymbol{w}$ is multiplicative; that is, $w_{i+j} = w_i w_j$ for all $i,j \in \mathbb{Z}$, or equivalently that $w_j = w_1^j$, $j \in \mathbb{Z}$. Under this extra assumption, we claim that the new modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ in Definition 1 enjoy a scaling symmetry. Indeed, by repeating the computation above, we obtain

$$\begin{aligned}\|f_\lambda\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} &= 2^{-j_0\frac{d}{p}} \Big\| \big\| \|\psi_{j-j_0,k}(D)f\|_{L^p_x(\mathbb{R}^d)} \big\|_{\ell^q_k(\mathbb{Z}^d)} \Big\|_{\ell^r_j(\boldsymbol{w})} \\ &= 2^{-j_0\frac{d}{p}} w_{j_0} \Big\| \big\| \|\psi_{j,k}(D)f\|_{L^p_x(\mathbb{R}^d)} \big\|_{\ell^q_k(\mathbb{Z}^d)} \Big\|_{\ell^r_j(\boldsymbol{w})} \\ &= \lambda^{\log_2(w_1)-\frac{d}{p}} \|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}}. \end{aligned} \tag{2.11}$$

This shows the scaling property of the $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$-spaces, when the vector weight $\boldsymbol{w}$ is multiplicative.

Remark 2.3. (i) The basic property of the Fourier transform states that modulations on the physical side correspond to translations on the Fourier side. Unit-scale modulations, that is, unit-scale translations on the Fourier side, yield the (usual) modulation spaces $M^{p,q}$ and the decomposition (2.3). On the other hand, the new modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ with scaling are generated by modulations of all dyadic scales, that is, by translations of all dyadic scales on the Fourier side. Recalling that a wavelet basis is generated by all dyadic dilations and translations of a given (nice) function on the physical side, it is natural that the family $\big\{\psi_{j,k}(\cdot) = \psi(2^{-j}\ \cdot\ -k)\big\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$, which is essentially a wavelet basis but on the Fourier side, appears in Definition 1.

(ii) As it is well known, the modulation spaces $M^{p,q}$ have an equivalent characterization via the short-time (or windowed) Fourier transform (STFT). Given a non-zero window function $\phi \in \mathcal{S}(\mathbb{R}^d)$, we define the STFT $V_\phi f$ of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to ϕ by[1]

$$V_\phi f(x,\xi) = \int_{\mathbb{R}^d} f(y)\overline{\phi(y-x)}e^{-2\pi i y\cdot\xi}\,dy.$$

Then, we have the equivalence of norms:

$$\|f\|_{M^{p,q}} \sim_\phi |||f|||_{M^{p,q}} := \|V_\phi f\|_{L^q_\xi L^p_x} = \big\|\|V_\phi f\|_{L^p_x}\big\|_{L^q_\xi},$$

where the implicit constants depend on the window function ϕ.

Given $j \in \mathbb{Z}$, let $\delta_j f(x) = 2^{-jd}f(2^{-j}x)$ and $\phi^j(x) = \phi(2^j x)$. Then, a direct computation (see (4.4) below) shows that

$$\|f\|_{M^{p,q}_{[j]}} = 2^{j\frac{d}{p'}}\|\delta_j f\|_{M^{p,q}} \sim_\phi 2^{j\frac{d}{p'}}|||\delta_j f|||_{M^{p,q}} = 2^{j\frac{d}{p'}}\|V_\phi(\delta_j f)\|_{L^q_\xi L^p_x}. \tag{2.12}$$

Moreover, a straightforward calculation shows that

$$\big(V_\phi(\delta_j f)\big)(x,\xi) = (V_{\phi^j}f)(2^{-j}x, 2^j\xi). \tag{2.13}$$

Then, from (2.12) and (2.13), we obtain

$$\|f\|_{M^{p,q}_{[j]}} \sim_\phi 2^{jd(\frac{1}{p'}+\frac{1}{p}-\frac{1}{q})}\|V_{\phi^j}f\|_{L^q_\xi L^p_x} = 2^{j\frac{d}{q'}}\|V_{\phi^j}f\|_{L^q_\xi L^p_x}.$$

Based on these considerations, if one prefers to use the $|||\cdot|||_{M^{p,q}}$-norm via the STFT to define the new modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ with scaling, the only difference will be in exchanging the vector weight $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ with the vector weight $\boldsymbol{\sigma} = \{2^{j\frac{d}{q'}}w_j\}_{j\in\mathbb{Z}}$. Namely,

[1] This is essentially the Wigner transform of f and ϕ.

if we define

$$|||f|||_{M^{p,q}_{[j]}} := \|V_{\phi^j} f\|_{L^q_\xi L^p_x},$$

then we obtain the following equivalence of norms:

$$\|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} \sim |||f|||_{\mathfrak{M}^{p,q,r}_{\boldsymbol{\sigma}}} := \big\| |||f|||_{M^{p,q}_{[j]}} \big\|_{\ell^r_j(\boldsymbol{\sigma})}.$$

(iii) The boundedness property of the dilation operator: $f(x) \mapsto f_\lambda(x) = f(\lambda x)$ on the modulation spaces was studied in [20]. In particular, it was shown in [20, Theorem 1.1] that there exist $c_1, c_2 > 0$, depending only on d, p, and q, such that

$$\lambda^{c_1} \|f\|_{M^{p,q}} \lesssim \|f_\lambda\|_{M^{p,q}} \lesssim \lambda^{c_2} \|f\|_{M^{p,q}} \tag{2.14}$$

for all $\lambda \geq 1$. When $0 < \lambda < 1$, the estimate (2.14) holds after switching the exponents c_1 and c_2. The point of our construction is that the modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ are not biased toward any particular scale. Moreover, when the vector weight is multiplicative, the $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$-spaces enjoy the *exact* dyadic scaling (2.11), which is relevant, for example, in the analysis of the nonlinear Schrödinger equation; see also Sections 3 and 4.

(iv) There are variants of the estimate (2.14) in the settings of weighted modulation spaces [4] or α-modulation spaces [15] which are interesting in their own right. Naturally, one could consider a similar construction of scale invariant modulation spaces in these contexts. We, however, do not pursue this issue here and leave it for interested readers.

(v) Let $1 \leq p_0 < \infty$. Then, for $p > p_0 > q' > 1$, the vector weight $\boldsymbol{w} = \boldsymbol{w}(p, p_0) = \{w_j\}_{j \in \mathbb{Z}}$ with

$$w_j = 2^{jd(\frac{1}{p} - \frac{1}{p_0})}$$

is a multiplicative weight satisfying (2.7) with $0 < \varepsilon < d \min\{\frac{1}{q'} - \frac{1}{p_0}, \frac{1}{p_0} - \frac{1}{p}\}$. Moreover, for this choice of weight, (2.11) implies that, for a fixed dyadic scale $\lambda = 2^{j_0}$,

$$\|f_\lambda\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} = \lambda^{-\frac{d}{p_0}} \|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}}.$$

Namely, $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}(p,p_0)}$ scales like $L^{p_0}(\mathbb{R}^d)$. See also Section 3.

3. A "good" weight and L^2-embedding

In the discussion below, we restrict our attention to the case $q = r$ and we simply write $\mathfrak{M}^{p,q,q}_{\boldsymbol{w}}$ as $\mathfrak{M}^{p,q}_{\boldsymbol{w}}$. We will show that, for a particular good vector weight $\boldsymbol{w}$ and $p, q > 2$, the space $\mathfrak{M}^{p,q}_{\boldsymbol{w}}$ is sufficiently large.

Given $p > 2$, fix a vector weight $\boldsymbol{w}(p) = \{w_j\}_{j \in \mathbb{Z}}$ with

$$w_j = 2^{jd\frac{p'-2}{2p'}}. \tag{3.1}$$

We see that $\boldsymbol{w}(p)$ is a multiplicative vector weight and is a "good" weight in the sense of the conditions (2.7), provided that $q > 2$.

Before proceeding further, let us recall the space $X_{p,q}$ defined in [2, p. 5260] via the norm:

$$\|f\|_{X_{p,q}} = \bigg(\sum_{j \in \mathbb{Z}} 2^{jd\frac{p-2}{2p}q} \sum_{k \in \mathbb{Z}^d} \|\psi_{j,k} f\|^q_{L^p} \bigg)^{\frac{1}{q}}. \tag{3.2}$$

The $X_{p,q}$-spaces appear in the improvement of the Strichartz estimates for the linear Schrödinger equation (see (4.8) below) and, in particular, play an important role in the study of the mass-critical nonlinear Schrödinger equations. See also [3, Proposition 2.1] and [19, Theorem 4.2] for the one-dimensional and two-dimensional versions of the $X_{p,q}$-spaces. Note that the vector weight $\{2^{jd\frac{p-2}{2p}}\}_{j\in\mathbb{Z}}$ appearing in (3.2) is precisely $\boldsymbol{w}(p')$ defined in (3.1).

Let us now come back to the $\mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}$-space with $\boldsymbol{w}(p) = \{w_j\}_{j\in\mathbb{Z}}$ defined in (3.1). For $p > 2$, it follows from Hausdorff-Young's inequality that

$$\begin{aligned}\|f\|_{\mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}} &= \bigg(\sum_{j\in\mathbb{Z}} w_j^q \sum_{k\in\mathbb{Z}^d} \|\psi_{j,k}(D)f\|^q_{L^p_x}\bigg)^{\frac{1}{q}}\\ &\le \bigg(\sum_{j\in\mathbb{Z}} 2^{jd\frac{p'-2}{2p'}q} \sum_{k\in\mathbb{Z}^d} \|\psi_{j,k}\widehat{f}\|^q_{L^{p'}}\bigg)^{\frac{1}{q}} = \|\widehat{f}\|_{X_{p',q}}\\ &=: \|f\|_{\mathcal{F}X_{p',q}}. \end{aligned} \tag{3.3}$$

This shows that

$$\mathcal{F}X_{p',q} \subset \mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}. \tag{3.4}$$

In particular, it follows from [2, Theorem 1.3] and Plancherel's theorem that for all $p, q > 2$, we have

$$\|f\|_{\mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}} \lesssim \|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R}^d)$. Namely, we have the following embedding:

$$L^2(\mathbb{R}^d) \subset \mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}(\mathbb{R}^d).$$

Moreover, this embedding is strict; indeed, if we set $f = \mathcal{F}^{-1}g$, where

$$g(\xi) = |\xi|^{-\frac{d}{2}}\big|\ln|\xi|\big|^{-\frac{1}{2}} \cdot \mathbf{1}_{(0,\frac{1}{2})^d}(\xi),$$

we see that $f \in \mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}(\mathbb{R}^d) \setminus L^2(\mathbb{R}^d)$ when $p, q > 2$; see again [2, p. 5267].

Remark 3.1. In the study of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces, the *Fourier-amalgam spaces* $\widehat{m}^{p,q}(\mathbb{R}^d)$, defined by the norm:

$$\|f\|_{\widehat{m}^{p,q}(\mathbb{R}^d)} = \big\|\|\psi(\xi-k)\widehat{f}\|_{L^p_\xi(\mathbb{R}^d)}\big\|_{\ell^q_k(\mathbb{Z}^d)},$$

were considered in [13]. We note that the Fourier-amalgam space $\widehat{m}^{p,q}(\mathbb{R}^d)$ is simply the Fourier image of the usual Wiener amalgam space $W(L^p,\ell^q)(\mathbb{R}^d)$, as described in [6, 14], defined by the norm:

$$\|f\|_{W(L^p,\ell^q)(\mathbb{R}^d)} = \big\|\|\psi(x-k)f\|_{L^p_x(\mathbb{R}^d)}\big\|_{\ell^q_k(\mathbb{Z}^d)}.$$

Comparing this with the definition (2.2) of the modulation spaces $M^{p,q}$, we see that $\widehat{m}^{p',q}$ "scales like" $M^{p,q}$ (in the sense that $L^{p'}_\xi(\mathbb{R}^d)$ scales in the same manner as $L^p_x(\mathbb{R}^d)$). In Definition 1, we defined the new modulation spaces $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ with scaling for a good vector weight $\boldsymbol{w}$ satisfying (2.7). In an analogous manner, we can also define the new Fourier-amalgam spaces $\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}$ adapted to scaling by the norm:

$$\|f\|_{\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}} := \big\|\|f\|_{\widehat{m}^{p,q}_{[j]}}\big\|_{\ell^r_j(\boldsymbol{w})} = \bigg(\sum_{j\in\mathbb{Z}} w_j^r \|f\|^r_{\widehat{m}^{p,q}_{[j]}}\bigg)^{\frac{1}{r}},$$

where the scaled Fourier-amalgam spaces $\widehat{m}^{p,q}_{[j]}$ are defined by

$$\|f\|_{\widehat{m}^{p,q}_{[j]}} := \big\| \|\psi_{j,k}\widehat{f}\|_{L^p_\xi(\mathbb{R}^d)} \big\|_{\ell^q_k(\mathbb{Z}^d)}.$$

As in the case of the $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$-spaces, we need to impose some conditions on the vector weight $\boldsymbol{w}$ so that the resulting space $\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}$ contains smooth functions. Recalling the scaling property of $\widehat{m}^{p,q}$, we impose the following condition on the vector weight $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$:

$$w_j \lesssim \begin{cases} 2^{-\varepsilon j} & \text{if } j \geq 0, \\ 2^{-(\frac{d}{p}-\frac{d}{q}-\varepsilon)j} & \text{if } j < 0. \end{cases} \tag{3.5}$$

Namely, we replace p' in (2.7) by p. Arguing as in Subsection 2.2, we see that this new Fourier-amalgam space $\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}$ enjoy a scaling property analogous to that for the $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$-spaces. In particular, if the vector weight $\boldsymbol{w}$ is multiplicative, then we have the following scaling property; if we let $\lambda = 2^{j_0}$ for some $j_0 \in \mathbb{Z}$ and set $f_\lambda(x) = f(\lambda x)$ as before, then

$$\|f_\lambda\|_{\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}} = \lambda^{\log_2(w_1)-\frac{d}{p'}}\|f\|_{\widehat{\mathfrak{m}}^{p,q,r}_{\boldsymbol{w}}}.$$

We conclude this remark by pointing out that the $\mathcal{F}X_{p,q}$-space discussed above is simply $\widehat{\mathfrak{m}}^{p,q,q}_{\boldsymbol{w}(p')}$ with the vector weight $\boldsymbol{w}(p') = \big\{2^{jd\frac{p-2}{2p}}\big\}_{j\in\mathbb{Z}}$. It is easy to see that this vector weight satisfies the condition (3.5) for $q > 2$ and it is also multiplicative, thus satisfying the following scaling property for $\lambda = 2^{j_0}$ for some $j_0 \in \mathbb{Z}$:

$$\begin{aligned} \|f_\lambda\|_{\mathcal{F}X_{p,q}} = \|f_\lambda\|_{\widehat{\mathfrak{m}}^{p,q,q}_{\boldsymbol{w}(p')}} &= \lambda^{d\frac{p-2}{2p}-\frac{d}{p'}}\|f\|_{\widehat{\mathfrak{m}}^{p,q,q}_{\boldsymbol{w}(p')}} \\ &= \lambda^{-\frac{d}{2}}\|f\|_{\widehat{\mathfrak{m}}^{p,q,q}_{\boldsymbol{w}}} = \lambda^{-\frac{d}{2}}\|f\|_{\mathcal{F}X_{p,q}}. \end{aligned}$$

Namely, the $\mathcal{F}X_{p,q}$-spaces (and hence the $X_{p,q}$-spaces) scale like $L^2(\mathbb{R}^d)$ as it is expected.

Remark 3.2. In studying the mass-subcritical generalized KdV equation, Masaki-Segata [18, Definition 2.1] considered the so-called *generalized Morrey spaces* $M^p_{q,r}$ (with $q \leq p$) and $\widehat{M}^p_{q,r} = \mathcal{F}M^{p'}_{q',r}$ (with $p \leq q$ and $r > p'$). The usual Morrey spaces M^p_q correspond to $r = \infty$ in the scale of generalized Morrey spaces, that is $M^p_q = M^p_{q,\infty}$. These spaces generalize $X_{p,q}$ and $\mathcal{F}X_{p',q}$ defined in (3.2) and (3.3) and in particular we have $X_{p,q} = M^2_{p,q}$ and $\mathcal{F}X_{p',q} = \widehat{M}^2_{p,q}$. We point out that, when $r < \infty$, the space $\widehat{M}^p_{q,r}$ is nothing but $\widehat{\mathfrak{m}}^{q',r,r}_{\boldsymbol{w}(p,q)}$ with a specific vector weight given by $\boldsymbol{w}(p,q) = \big\{2^{jd(\frac{1}{q}-\frac{1}{p})}\big\}_{j\in\mathbb{Z}}$,[2] while $M^p_{q,r}$ is the Fourier image of $\widehat{\mathfrak{m}}^{q,r,r}_{\boldsymbol{w}'(p,q)}$ with the vector weight $\boldsymbol{w}'(p,q) = \big\{2^{jd(\frac{1}{p}-\frac{1}{q})}\big\}_{j\in\mathbb{Z}}$. For more on Morrey-type spaces, see also [10, 11, 12].

4. An application to the Schrödinger multiplier

Let $S(t) = e^{-it\Delta}$ be the Schrödinger operator defined as a Fourier multiplier operator with a multiplier $e^{4\pi^2 it|\xi|^2}$. Our main interest here is to discuss a boundedness property of the Schrödinger operator $S(t) = e^{-it\Delta}$ on appropriate $\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$-spaces.

[2] Under $r > p'$, this vector weight satisfies the condition (3.5) for $q < 2$.

Let ψ be as in (2.1). A direct computation shows that

$$\begin{aligned}\psi_{j,k}(D)(S(t)f)(x) &= \int_{\mathbb{R}^d} \psi(2^{-j}\xi - k)e^{4\pi^2 it|\xi|^2}\widehat{f}(\xi)e^{2\pi i x\cdot\xi}\,d\xi \\ &= 2^{jd}\int_{\mathbb{R}^d} \psi(\xi - k)e^{4\pi^2 i(2^{2j}t)|\xi|^2}\widehat{f}(2^j\xi)e^{2\pi i 2^j x\cdot\xi}\,d\xi \\ &= 2^{jd}\big(\psi(D-k)(S(2^{2j}t)\delta_j f)\big)(2^j x), \end{aligned} \tag{4.1}$$

where, as before, we wrote δ_j for the dilation operator given by $\delta_j f(x) = 2^{-jd}f(2^{-j}x)$. Thus, we have

$$\|\psi_{j,k}(D)(S(t)f)\|_{L^p_x} = 2^{j\frac{d}{p'}}\|\psi(D-k)(S(2^{2j}t)\delta_j f)\|_{L^p_x}.$$

Hence, we obtain

$$\|S(t)f\|_{M^{p,q}_{[j]}} = 2^{j\frac{d}{p'}}\|S(2^{2j}t)\delta_j f\|_{M^{p,q}}. \tag{4.2}$$

Interpolating [1, Theorem 14] for $p = 1$ or $p = \infty$ with the trivial bound for $p = 2$, we obtain

$$\|S(2^{2j}t)\delta_j f\|_{M^{p,q}} \lesssim \langle 2^{2j}t\rangle^{d\,|\frac{1}{2}-\frac{1}{p}|}\|\delta_j f\|_{M^{p,q}}. \tag{4.3}$$

Lastly, by a computation analogous to (4.1), we have

$$\psi(D-k)(\delta_j f)(x) = 2^{-jd}\psi_{j,k}(D)f(2^{-j}x)$$

and hence

$$\|\delta_j f\|_{M^{p,q}} = 2^{-jd}2^{j\frac{d}{p}}\big\|\|\psi_{j,k}(D)f\|_{L^p_x}\big\|_{\ell^q_k} = 2^{-j\frac{d}{p'}}\|f\|_{M^{p,q}_{[j]}}. \tag{4.4}$$

Putting (4.2), (4.3), and (4.4) together, we obtain

$$\|S(t)f\|_{M^{p,q}_{[j]}} \lesssim \langle 2^{2j}t\rangle^{d\,|\frac{1}{2}-\frac{1}{p}|}\|f\|_{M^{p,q}_{[j]}}. \tag{4.5}$$

Note that, when $j = 0$, this estimate recovers the boundedness of $S(t)$ on the modulation spaces $M^{p,q}$. We also point out that, when $p \neq 2$, the divergent behavior of the constant as $j \to \infty$ is consistent with the unboundedness of $S(t)$ on $L^p(\mathbb{R}^d)$.

Now, let $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ be a "good" vector weight, satisfying (2.7), and define a vector weight $\boldsymbol{\sigma} = \{\sigma_j\}_{j\in\mathbb{Z}}$ by setting

$$\sigma_j = \begin{cases} 2^{-jd\,|1-\frac{2}{p}|}w_j & \text{if } j \geq 0, \\ w_j & \text{if } j < 0. \end{cases}$$

Note that such $\boldsymbol{\sigma}$ also satisfies (2.7). Then, from (4.5), we obtain the boundedness of $S(t): \mathfrak{M}^{p,q,r}_{\boldsymbol{w}} \to \mathfrak{M}^{p,q,r}_{\boldsymbol{\sigma}}$ with a quantitative bound:

$$\|S(t)f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{\sigma}}} \lesssim \langle t\rangle^{d\,|\frac{1}{2}-\frac{1}{p}|}\|f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}}$$

for any $t \in \mathbb{R}$.

Let us further comment on an alternate estimate when $p \geq 2$. In this case, the Schrödinger operator is known to also satisfy the following inequality (see [16, Proposition 4.1]):

$$\|S(t)f\|_{M^{p,q}} \lesssim \langle t\rangle^{-d\,(\frac{1}{2}-\frac{1}{p})}\|f\|_{M^{p',q}}.$$

Note that the decay at infinity in this estimate is now consistent with the one in the $L^{p'} \to L^p$ boundedness of $S(t)$. Thus, instead of (4.3), we can now write

$$\|S(2^{2j}t)\delta_j f\|_{M^{p,q}} \lesssim \langle 2^{2j}t\rangle^{d\,(\frac{1}{2}-\frac{1}{p})}\|\delta_j f\|_{M^{p',q}}. \tag{4.6}$$

Since, by (4.4), we have $\|\delta_j f\|_{M^{p',q}} = 2^{-j\frac{d}{p}}\|f\|_{M^{p',q}_{[j]}}$, (4.2) and (4.6) yield

$$\begin{aligned}\|S(t)f\|_{M^{p,q}_{[j]}} &\lesssim \langle 2^{2j}t\rangle^{-d\,(\frac{1}{2}-\frac{1}{p})}2^{jd\,(\frac{1}{p'}-\frac{1}{p})}\|f\|_{M^{p',q}_{[j]}}\\ &= \langle 2^{2j}t\rangle^{-d\,(\frac{1}{2}-\frac{1}{p})}2^{2jd\,(\frac{1}{2}-\frac{1}{p})}\|f\|_{M^{p',q}_{[j]}}.\end{aligned} \tag{4.7}$$

Thus, by considering again $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{Z}}$ to be a "good" vector weight, satisfying (2.7), from (4.7) we obtain now the boundedness $S(t) : \mathfrak{M}^{p',q,r}_{\boldsymbol{w}} \to \mathfrak{M}^{p,q,r}_{\boldsymbol{w}}$ with a quantitative bound:

$$\|S(t)f\|_{\mathfrak{M}^{p,q,r}_{\boldsymbol{w}}} \lesssim \langle t\rangle^{-d\,(\frac{1}{2}-\frac{1}{p})}\|f\|_{\mathfrak{M}^{p',q,r}_{\boldsymbol{w}}}$$

for any $p \geq 2$ and $t \in \mathbb{R}$.

We conclude our note with a comment on the following improvement of the Strichartz estimate proved in [2, Theorem 1.2]; if $2 < p < 2 + \frac{4}{d(d+3)}$ and $q = \frac{2(d+2)}{d}$, then

$$\|S(t)f\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{F}X_{p',q}(\mathbb{R}^d)}. \tag{4.8}$$

The proof of this inequality is highly non-trivial; it uses the bilinear restriction estimate in [21, Theorem 1.1] and an appropriate orthogonality lemma for functions with disjoint Fourier supports [22, Lemma 6.1]. Considering the embedding (3.4) stated in the previous section, it would be interesting to know if the $\mathcal{F}X_{p',q}$-norm in (4.8) can be replaced by the $\mathfrak{M}^{p,q}_{\boldsymbol{w}(p)}$-norm.

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